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"NATURAL", "KINEMATIC" AND "ELASTIC" DISPLACEMENTS OF UNDERCONSTRAINED STRUCTURES

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Abstract—Analysis of pin-bars assemblies in which the number of degrees of freedom is greater than equilibrium matrix rank (underconstrained structures) is carried out. Some approaches to the linear analysis of statics, stability and vibrations are presented. It is shown that the main feature of underconstrained structures is a dominant role of initial equilibrium state or prestressing.

1. INTRODUCTION

Various methods were proposed for analysis of underconstrained structures. According to the Pellegrino–Calladine (Pellegrino and Calladine, 1986) concept the displacements of the underconstrained structures are resolved into "inextensional mechanism" displacements and "displacements which require elongations". Kuznetsov (1991) considered analysis of underconstrained structures from the analytical mechanics point of view. He proposed a new classification of structures which extends classical concepts. Vilnay (1990) dealt with the problems of design, and statical and dynamical response of reticulated underconstrained structures. According to his approach displacements of underconstrained structures are resolved into "small" and "large" components. The referred publications contain a wide bibliography on the problem. The present paper is an attempt to expound linear analysis of underconstrained structures as a whole from the traditional structural mechanics point of view. Clarification of the features of analysis of underconstrained structures is attained by neglecting small terms in the classical structural mechanics equations. The paper investigates assumptions and tactics used by different methods proposed for analysis of underconstrained structures as related to the well established theory of structures. It includes previous analysis and some new developments.

Arbitrary assembly of pin-jointed bars is in equilibrium if the following statical condition is satisfied

$$\mathbf{A}_0\mathbf{P}_0 = \mathbf{Q}_0 \tag{1}$$

$\mathbf{A}_0$ is m by n initial configuration equilibrium matrix, $\mathbf{P}_0$ is n by 1 column matrix of initial member forces, $\mathbf{Q}_0$ is m by 1 column matrix of initial external loads. In the particular case where $\mathbf{Q}_0 = 0$ the self stress state for the nontrivial solution of eqn (1) is obtained.

For simplicity, initial elongations and displacements are set to zero, which generally indicates origin reference but not physical absence of deformations. In fact the deformed configuration is considered as the initial one.

Under external load $\mathbf{Q}$, eqn (1) has to be replaced by the following

$$(\mathbf{A}_0+\mathbf{A})(\mathbf{P}_0+\mathbf{P}) = \mathbf{Q}_0+\mathbf{Q} \tag{2}$$

in which the values without indices are increments of the corresponding initial values. By using eqn (1) and linearizing eqn (2) the last one takes the form

$$\mathbf{A}_0\mathbf{P}+\mathbf{A}\mathbf{P}_0 = \mathbf{Q}. \tag{3}$$

The kinematic equation takes the form

$$\mathbf{A}_0^{\mathrm{T}}\mathbf{U} = \Delta \tag{4}$$

where Δ is n by 1 column matrix of member elongations and $\mathbf{U}$ is m by 1 column matrix of nodal displacements.

By adding Hooke's law

$$\mathbf{P} = \mathbf{S}\Delta \tag{5}$$

where $\mathbf{S}$ is the uncoupled stiffness matrix; a closed system of eqns (3)–(5) for unknowns $\mathbf{U}$, $\mathbf{P}$, Δ is obtained.

Since, after the linearization, elements of the perturbated equilibrium matrix $\mathbf{A}$ depend linearly upon nodal displacements, the second term on the left-hand side of eqn (3) takes the form

$$\mathbf{A}\mathbf{P}_0 = \mathbf{D}\mathbf{U}$$
$$\mathbf{D} \equiv \mathbf{D}(\mathbf{P}_0) \tag{6}$$

where $\mathbf{D}$ is m by m matrix whose elements are linear combinations of components of the initial member forces $\mathbf{P}_0$.

By using eqns (4)–(6), eqn (3) takes the form

$$\mathbf{K}\mathbf{U} = \mathbf{Q} \tag{7}$$

$$\mathbf{K} = \mathbf{A}_0\mathbf{S}\mathbf{A}_0^{\mathrm{T}}+\mathbf{D} \tag{8}$$

where $\mathbf{K}$ is m by m stiffness matrix. $\mathbf{K}$ must be positive definite to provide stability of the equilibrium state. Physically it means that increments of external forces produce increments of nodal displacements in the external forces directions. Equation (7) is a matrix formulation of pin-jointed bars assemblies analysis in terms of displacements. It may be employed in the analysis of all possible types of structures. Nevertheless, it is preferable to simplify the analysis by neglecting small values relating to the discussed structure.

It is possible to define two global classes of structures. The first class can be defined as traditional (conventional, fully constrained). Its formal indication is

$$m = r = \operatorname{rank}\mathbf{A}_0. \tag{9}$$

In this case the second term on the right-hand side of eqn (8) can be neglected (this assumption seems to be valid for most construction materials) and the analysis follows the wellknown traditional method of analysis of fully constrained reticulated structures.

The second class of structures can be defined as nontraditional (nonconventional, underconstrained). Its formal indication is

$$m > r. \tag{10}$$

In this case the second term on the right-hand side of eqn (8) can not be neglected, although its elements are less than elements of the first one, since rank $(\mathbf{A}_0\mathbf{S}\mathbf{A}_0^{\mathrm{T}}) < m$. On the other hand, direct solution of eqn (7) is undesirable because matrix $\mathbf{K}$ is ill-conditioned.

2. THE "NATURAL" APPROACH OF THE RESOLUTION OF DISPLACEMENTS

In the case of underconstrained structures the kinematic equation (4) contains more unknowns than there are scalar equations. Consequently, its solution may be presented in the form

$$\mathbf{U} = \mathbf{U}^{h} + \mathbf{U}^{p} \tag{11}$$

where $\mathbf{U}^{h}$ and $\mathbf{U}^{p}$ are the solution of the homogeneous equation (4), and the particular solution of eqn (4), respectively.

In the general case, the modes of $\mathbf{U}^{p}$ and consequently $\mathbf{U}^{h}$ depend upon a computational scheme. If Gaussian elimination procedure is used then it is natural to present displacements in the form

$$\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1^{h} \\ \mathbf{U}_2^{h} \end{bmatrix} + \begin{bmatrix} \mathbf{U}_1^{p} \\ \mathbf{0} \end{bmatrix} \tag{12}$$

where $\mathbf{U}_1$, $\mathbf{U}_1^{h}, \mathbf{U}_1^{p}$ are r by 1 column matrices and $\mathbf{U}_2$, $\mathbf{U}_2^{h}$ are $m-r$ by 1 column matrices. This resolution is made in accordance with the matrix $\mathbf{A}_0$ resolution

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_{011}(r \times r) & \mathbf{A}_{012}(r \times n-r) \\ \mathbf{A}_{021}(m-r \times r) & \mathbf{A}_{022}(m-r \times n-r) \end{bmatrix} \tag{13}$$

where submatrices' dimensions are shown in parentheses.

By definition

$$\det \mathbf{A}_{011} \neq 0. \tag{14}$$

Since the rows of the two lower submatrices in eqn (13) are linear combinations of the upper submatrices' rows then

$$\mathbf{A}_{021} = \mathbf{T}\mathbf{A}_{011}, \quad \mathbf{A}_{022} = \mathbf{T}\mathbf{A}_{012}. \tag{15}$$

Here $\mathbf{T}$ is $m-r$ by r matrix.

By substituting resolutions (12) and (13) into homogeneous eqn (4) and taking into account eqns (15) it is obtained that

$$\mathbf{U}_1^{h} = -\mathbf{T}^{T}\mathbf{U}_2^{h}. \tag{16}$$

Consequently

$$\begin{aligned} \mathbf{U}_1 &= \mathbf{U}_1^{p} - \mathbf{T}^{T}\mathbf{U}_2^{h} \\ \mathbf{U}_2 &= \mathbf{U}_2^{h}. \end{aligned} \tag{17}$$

Further considerations require corresponding resolutions of used matrices and vectors

$$\mathbf{D}=\begin{bmatrix}\mathbf{D}_{11}(r\times r) & \mathbf{D}_{12}(r\times m-r)\\ \mathbf{D}_{21}(m-r\times r) & \mathbf{D}_{22}(m-r\times m-r)\end{bmatrix},\quad \mathbf{S}=\begin{bmatrix}\mathbf{S}_{11}(r\times r) & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_{22}(n-r\times n-r)\end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix}\mathbf{Q}_1(r\times 1)\\ \mathbf{Q}_2(m-r\times 1)\end{bmatrix},\quad \mathbf{P}_0=\begin{bmatrix}\mathbf{P}_{01}(r\times 1)\\ \mathbf{P}_{02}(n-r\times 1)\end{bmatrix},\quad \mathbf{P}=\begin{bmatrix}\mathbf{P}_1(r\times 1)\\ \mathbf{P}_2(n-r\times 1)\end{bmatrix},\quad \Delta=\begin{bmatrix}\Delta_1(r\times 1)\\ \Delta_2(n-r\times 1)\end{bmatrix}.$$

2.1. *Statics*

Initial equilibrium equations may be written in the form (initial external load is absent)

$$\begin{aligned}\mathbf{A}_{011}\mathbf{P}_{01}+\mathbf{A}_{012}\mathbf{P}_{02}&=\mathbf{0}\\ \mathbf{A}_{021}\mathbf{P}_{01}+\mathbf{A}_{022}\mathbf{P}_{02}&=\mathbf{0}.\end{aligned} \tag{18}$$

By solving eqns (18) taking into account eqns (15) it is obtained that

$$\mathbf{P}_{01} = -\mathbf{A}_{011}^{-1}\mathbf{A}_{012}\mathbf{P}_{02}. \tag{19}$$

Thus, $\mathbf{P}_{02}$ is a column matrix of given prestressing forces.

By using eqn (6) and considered resolutions, eqn (3) takes the form

$$\begin{aligned}\mathbf{A}_{011}\mathbf{P}_1+\mathbf{A}_{012}\mathbf{P}_2+\mathbf{D}_{11}\mathbf{U}_1+\mathbf{D}_{12}\mathbf{U}_2&=\mathbf{Q}_1\\ \mathbf{A}_{021}\mathbf{P}_1+\mathbf{A}_{022}\mathbf{P}_2+\mathbf{D}_{21}\mathbf{U}_1+\mathbf{D}_{22}\mathbf{U}_2&=\mathbf{Q}_2.\end{aligned} \tag{20}$$

By subtracting the product of $\mathbf{T}$ and the first equation from the second one, and taking into account eqns (15), eqns (20) take the form

$$\begin{aligned}\mathbf{A}_{011}\mathbf{P}_1+\mathbf{A}_{012}\mathbf{P}_2+\mathbf{D}_{11}\mathbf{U}_1+\mathbf{D}_{12}\mathbf{U}_2&=\mathbf{Q}_1\\ (\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{U}_1+(\mathbf{D}_{22}-\mathbf{T}\mathbf{D}_{12})\mathbf{U}_2&=\mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1.\end{aligned} \tag{21}$$

Here

$$\begin{aligned}\mathbf{P}_1&=\mathbf{S}_{11}\mathbf{A}_{011}^{\mathrm{T}}\mathbf{U}_1+\mathbf{S}_{11}\mathbf{A}_{021}^{\mathrm{T}}\mathbf{U}_2\\ \mathbf{P}_2&=\mathbf{S}_{22}\mathbf{A}_{012}^{\mathrm{T}}\mathbf{U}_1+\mathbf{S}_{22}\mathbf{A}_{022}^{\mathrm{T}}\mathbf{U}_2.\end{aligned} \tag{22}$$

By using eqns (17) and (22), eqns (21) take the form

$$(\mathbf{K}_1+\mathbf{D}_{11})\mathbf{U}_1^{\mathrm{p}}+(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^{\mathrm{T}})\mathbf{U}_2^{\mathrm{h}}=\mathbf{Q}_1 \tag{23}$$

$$(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{U}_1^{\mathrm{p}}+\mathbf{K}_2\mathbf{U}_2^{\mathrm{h}}=\mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1 \tag{24}$$

where

$$\mathbf{K}_1=\mathbf{A}_{011}\mathbf{S}_{11}\mathbf{A}_{011}^{\mathrm{T}}+\mathbf{A}_{012}\mathbf{S}_{22}\mathbf{A}_{012}^{\mathrm{T}} \tag{25}$$

$$\mathbf{K}_2=\mathbf{D}_{22}+\mathbf{T}\mathbf{D}_{11}\mathbf{T}^{\mathrm{T}}-\mathbf{T}\mathbf{D}_{12}-\mathbf{D}_{21}\mathbf{T}^{\mathrm{T}}. \tag{26}$$

After some transformations eqns (23) and (24) take the form

$$\{\mathbf{K}_1-(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^{\mathrm{T}})\mathbf{K}_2^{-1}(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\}\mathbf{U}_1^{\mathrm{p}}=\mathbf{Q}_1-(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^{\mathrm{T}})\mathbf{K}_2^{-1}(\mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1) \tag{27}$$

$$\{\mathbf{K}_2-(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{K}_1^{-1}(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^{\mathrm{T}})\}\mathbf{U}_2^{\mathrm{h}}=\mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1-(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{K}_1^{-1}\mathbf{Q}_1. \tag{28}$$

Since the elements of matrices $\mathbf{K}_1$, $\mathbf{S}$ are significantly larger than elements of matrices $\mathbf{K}_2$, $\mathbf{D}_{ij}$, $\mathbf{P}_0$:

$$\|\mathbf{K}_1\| \approx \|\mathbf{S}\| \gg \|\mathbf{K}_2\| \approx \|\mathbf{D}_{ij}\| \approx \|\mathbf{P}_0\| \tag{29}$$

then $\mathbf{K}_1+\mathbf{D}_{11} \cong \mathbf{K}_1$, and by neglecting the second terms in braces of eqns (27) and (28) in comparison with the first ones it is possible to obtain that

$$\mathbf{K}_1\mathbf{U}_1^p = \mathbf{F}_1 \tag{30}$$

$$\mathbf{K}_2\mathbf{U}_2^h = \mathbf{F}_2 \tag{31}$$

where

$$\mathbf{F}_1 = \mathbf{Q}_1-(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^T)\mathbf{K}_2^{-1}(\mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1) \tag{32}$$

$$\mathbf{F}_2 = \mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1-(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{K}_1^{-1}\mathbf{Q}_1. \tag{33}$$

Finally, the member forces take the form

$$\begin{aligned}\mathbf{P}_1 &= \mathbf{S}_{11}\mathbf{A}_{011}^T\mathbf{K}_1^{-1}\mathbf{F}_1\\ \mathbf{P}_2 &= \mathbf{S}_{22}\mathbf{A}_{012}^T\mathbf{K}_1^{-1}\mathbf{F}_1.\end{aligned} \tag{34}$$

It is necessary to note that the equilibrium state is stable only if both matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ are positive definite.

Matrix $\mathbf{K}_1$ is affected by elastic properties of the structure and it is positive definite due to its framework. [Obviously, everything discussed is true if $\det(\mathbf{K}_1+\mathbf{D}_{11}) \neq 0$, otherwise the initial equilibrium state is unstable].

Matrix $\mathbf{K}_2$ is affected by initial member forces and its positiveness must be provided under design of initial equilibrium state.

2.2. *Instability: bifurcation of equilibrium*

The structure is in equilibrium if eqn (2) is satisfied. In the case of the bifurcation of the equilibrium state nonzero perturbations $\mathbf{A}'$, $\mathbf{P}'$ exist

$$(\mathbf{A}_0+\mathbf{A}+\mathbf{A}')(\mathbf{P}_0+\mathbf{P}+\mathbf{P}') = \mathbf{Q}_0+\mathbf{Q} \tag{35}$$

or expanding eqn (35) and omitting small values

$$\mathbf{A}_0\mathbf{P}'+\mathbf{A}'(\mathbf{P}_0+\mathbf{P})=0 \tag{36}$$

where

$$\mathbf{P}' = \mathbf{S}\mathbf{A}_0^T\,\delta\mathbf{U} \tag{37}$$

$$\begin{aligned}\mathbf{A}'(\mathbf{P}_0+\mathbf{P}) &= \mathbf{D}'\delta\mathbf{U}\\ \mathbf{D}' &\equiv \mathbf{D}(\mathbf{P}_0)+\mathbf{D}(\mathbf{P}).\end{aligned} \tag{38}$$

By this means, equilibrium equations in terms of displacements can be obtained directly from eqns (23) and (24) with the help of replacements

$$\mathbf{U}_1^p \to \delta\mathbf{U}_1^p,\quad \mathbf{U}_2^h \to \delta\mathbf{U}_2^h,\quad \mathbf{D}_{ij} \to \mathbf{D}'_{ij},\quad \mathbf{Q}_i \to 0.$$

The following results from eqns (23) and (24)

$$(\mathbf{K}_1+\mathbf{D}'_{11})\delta\mathbf{U}_1^p+(\mathbf{D}'_{12}-\mathbf{D}'_{11}\mathbf{T}^T)\delta\mathbf{U}_2^h = 0 \tag{39}$$

$$(\mathbf{D}'_{21}-\mathbf{T}\mathbf{D}'_{11})\delta\mathbf{U}_1^p+\mathbf{K}'_2\delta\mathbf{U}_2^h = 0 \tag{40}$$

where $\mathbf{K}_1$ is defined in eqn (25) and

$$\mathbf{K}'_2 = \mathbf{D}'_{22}+\mathbf{T}\mathbf{D}'_{11}\mathbf{T}^T-\mathbf{T}\mathbf{D}'_{12}-\mathbf{D}'_{21}\mathbf{T}^T. \tag{41}$$

If the external load $\mathbf{Q}$ grows proportionally to parameter ν then member forces may be presented in the form $\nu\mathbf{P}$ [see eqns (34)]. Consequently

$$\mathbf{D}' \equiv \mathbf{D}(\mathbf{P}_0)+\nu\mathbf{D}(\mathbf{P}).$$

Nonzero displacements appear under such values of parameter ν which lead to the equality

$$\det\begin{bmatrix} \mathbf{K}_1+\mathbf{D}'_{11} & \mathbf{D}'_{12}\mathbf{T}^T \\ \mathbf{D}'_{21}-\mathbf{T}\mathbf{D}'_{11} & \mathbf{K}'_2 \end{bmatrix} = 0. \tag{42}$$

Because of assumption (29) the left upper submatrix elements are significantly larger than the rest. By using Laplace's resolution of determinant it is possible to approximately replace eqn (42) by the following one

$$\det(\mathbf{K}_1+\mathbf{D}'_{11})\det\mathbf{K}'_2 = 0. \tag{43}$$

Again due to assumption (29) it is obvious that the minimum value of ν may be obtained from the equation

$$\det\mathbf{K}'_2 = 0 \tag{44}$$

or

$$\det\left\{\begin{matrix}[\mathbf{D}_{22}(\mathbf{P}_0)+\mathbf{T}\mathbf{D}_{11}(\mathbf{P}_0)\mathbf{T}^T-\mathbf{T}\mathbf{D}_{12}(\mathbf{P}_0)-\mathbf{D}_{21}(\mathbf{P}_0)\mathbf{T}^T]+ \\ +\nu[\mathbf{D}_{22}(\mathbf{P})+\mathbf{T}\mathbf{D}_{11}(\mathbf{P})\mathbf{T}^T-\mathbf{T}\mathbf{D}_{12}(\mathbf{P})-\mathbf{D}_{21}(\mathbf{P})\mathbf{T}^T]\end{matrix}\right\} = 0. \tag{45}$$

2.3. *Vibrations*

In the case of vibrations, equilibrium equations (23) and (24) take the form

$$(\mathbf{K}_1+\mathbf{D}_{11})\mathbf{U}_1^p+(\mathbf{D}_{12}-\mathbf{D}_{11}\mathbf{T}^T)\mathbf{U}_2^h+\mathbf{M}_{11}\ddot{\mathbf{U}}_1^p-\mathbf{M}_{11}\mathbf{T}^T\ddot{\mathbf{U}}_2^h = \mathbf{Q}_1 \tag{46}$$

$$(\mathbf{D}_{21}-\mathbf{T}\mathbf{D}_{11})\mathbf{U}_1^p+\mathbf{K}_2\mathbf{U}_2^h-\mathbf{T}\mathbf{M}_{11}\ddot{\mathbf{U}}_1^p+(\mathbf{M}_{22}+\mathbf{T}\mathbf{M}_{11}\mathbf{T}^T)\ddot{\mathbf{U}}_2^h = \mathbf{Q}_2-\mathbf{T}\mathbf{Q}_1 \tag{47}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11}(r\times r) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{22}(m-r\times m-r) \end{bmatrix}$$

is the matrix of masses.

Let the external load be presented in the form

$$\begin{aligned}\mathbf{Q}_1 &= \bar{\mathbf{Q}}_1\sin(\omega t+\vartheta) \\ \mathbf{Q}_2 &= \bar{\mathbf{Q}}_2\sin(\omega t+\vartheta)\end{aligned} \tag{48}$$

then

$$\mathbf{U}_1^p = \bar{\mathbf{U}}_1^p \sin(\omega t + \vartheta)$$

$$\mathbf{U}_2^h = \bar{\mathbf{U}}_2^h \sin(\omega t + \vartheta) \tag{49}$$

where a bar over a letter designates amplitude value, ω is angular frequency, ϑ is initial phase of vibrations.

Then eqns (46) and (47) take the form

$$\mathbf{L}_{11}\bar{\mathbf{U}}_1^p + \mathbf{L}_{12}\bar{\mathbf{U}}_2^h = \bar{\mathbf{Q}}_1 \tag{50}$$

$$\mathbf{L}_{21}\bar{\mathbf{U}}_1^p + \mathbf{L}_{22}\bar{\mathbf{U}}_2^h = \bar{\mathbf{Q}}_2 - \mathbf{T}\bar{\mathbf{Q}}_1 \tag{51}$$

where

$$\mathbf{L}_{11} = \mathbf{K}_1 + \mathbf{D}_{11} - \omega^2 \mathbf{M}_{11} \tag{52}$$

$$\mathbf{L}_{12} = \mathbf{D}_{12} - \mathbf{D}_{11}\mathbf{T}^T + \omega^2 \mathbf{M}_{11}\mathbf{T}^T \tag{53}$$

$$\mathbf{L}_{21} = \mathbf{D}_{21} - \mathbf{T}\mathbf{D}_{11} + \omega^2 \mathbf{T}\mathbf{M}_{11} \tag{54}$$

$$\mathbf{L}_{22} = \mathbf{K}_2 - \omega^2(\mathbf{M}_{22} + \mathbf{T}\mathbf{M}_{11}\mathbf{T}^T). \tag{55}$$

In the case of free vibrations $\bar{\mathbf{Q}}_1 = \bar{\mathbf{Q}}_2 = 0$ and corresponding frequencies are obtained from equation

$$\det\begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} = 0. \tag{56}$$

Since $\| \mathbf{L}_{11} \| \gg \| \mathbf{L}_{12} \| \approx \| \mathbf{L}_{21} \| \approx \| \mathbf{L}_{22} \|$ then by using Laplace's resolution of the determinant eqn (56) takes the form

$$\det \mathbf{L}_{11} \det \mathbf{L}_{22} = 0. \tag{57}$$

Because of assumption (29) the low frequencies are obtained using condition

$$\det \mathbf{L}_{22} = \det[\mathbf{K}_2 - \omega^2(\mathbf{M}_{22} + \mathbf{T}\mathbf{M}_{11}\mathbf{T}^T)] = 0 \tag{58}$$

and the high frequencies are obtained using condition

$$\det \mathbf{L}_{11} = \det(\mathbf{K}_1 - \omega^2 \mathbf{M}_{11} + \mathbf{D}_{11}) = 0. \tag{59}$$

If forced vibrations frequency coincides with the free one then displacements grow up unlimitedly. Otherwise displacements values are obtained by using equations

$$(\mathbf{L}_{11} - \mathbf{L}_{12}\mathbf{L}_{22}^{-1}\mathbf{L}_{21})\bar{\mathbf{U}}_1^p = \bar{\mathbf{Q}}_1 - \mathbf{L}_{12}\mathbf{L}_{22}^{-1}(\bar{\mathbf{Q}}_2 - \mathbf{T}\bar{\mathbf{Q}}_1) \tag{60}$$

$$(\mathbf{L}_{22} - \mathbf{L}_{21}\mathbf{L}_{11}^{-1}\mathbf{L}_{12})\bar{\mathbf{U}}_2^h = \bar{\mathbf{Q}}_2 - \mathbf{T}\bar{\mathbf{Q}}_1 - \mathbf{L}_{21}\mathbf{L}_{11}^{-1}\bar{\mathbf{Q}}_1 \tag{61}$$

or approximately

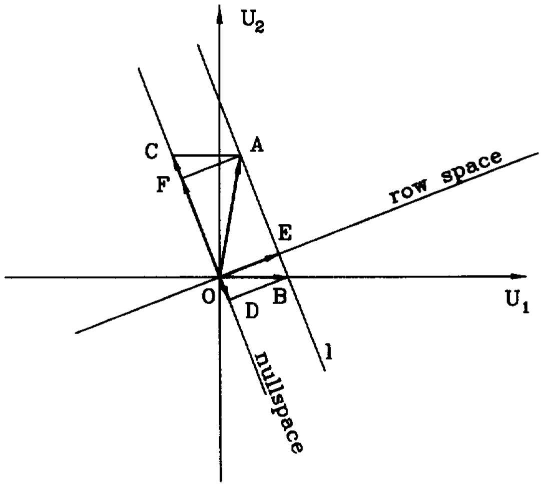


Fig. 1. Resolution of the displacements' modes.

$$\bar{\mathbf{U}}_1^p = \mathbf{L}_{22}^{-1}\{\bar{\mathbf{Q}}_1 - \mathbf{L}_{12}\mathbf{L}_{22}^{-1}(\bar{\mathbf{Q}}_2 - \mathbf{T}\bar{\mathbf{Q}}_1)\} \tag{62}$$

$$\bar{\mathbf{U}}_2^h = \mathbf{L}_{11}^{-1}\{\bar{\mathbf{Q}}_2 - \mathbf{T}\bar{\mathbf{Q}}_1 - \mathbf{L}_{21}\mathbf{L}_{11}^{-1}\bar{\mathbf{Q}}_1\}. \tag{63}$$

2.4. *Geometrical interpretation of displacements*

It is possible to consider displacements column matrix as a vector of m-dimensional Euclidian space $\Re^m$. Consider $\Re^2$ for the following reasons illustrated in Fig. 1.

The general solution of the kinematic equation (4) is presented by straight line l in Fig. 1. It is necessary to take into account equilibrium equations and Hooke's low to fix a unique solution. Let point A be a solution of structural mechanics problem and vector $\overrightarrow{OA}$ be the displacements vector

$$\overrightarrow{OA} = \mathbf{U}.$$

In accordance with resolution (11) this vector is presented in the form

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$$
$$\overrightarrow{OC} = \mathbf{U}^h$$
$$\overrightarrow{OB} = \mathbf{U}^p.$$

Vector $\overrightarrow{OC}$ belongs to the matrix $\mathbf{A}_0^T$ nullspace which is parallel to the line l (Strang, 1988). Vector $\overrightarrow{OB}$, by definition, coincides with the horizontal axis.

3. THE "KINEMATIC/ELASTIC" APPROACH OF THE RESOLUTION OF DISPLACEMENTS

As it is shown above vector $\overrightarrow{OC}$ belongs to the nullspace (it is the solution of homogeneous equations), then it has a clear physical sense. Namely, displacements associated with it do not produce elongations of the members.

Vector $\overrightarrow{OB}$ may be presented as a sum of two vectors

$$\overrightarrow{OB} = \overrightarrow{OD} + \overrightarrow{OE}$$

where vector $\overrightarrow{OD}$ belongs to the nullspace and vector $\overrightarrow{OE}$ is orthogonal to the last one.

Thus, the particular solution $\mathbf{U}^p$ partly contains displacements ($\overrightarrow{OD}$) which do not produce elongations. By summing all displacements without elongations, purely "kinematic" displacements $\mathbf{U}^k$ are obtained

$$\mathbf{U}^k = \overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{OD}.$$

Obviously vector $\overrightarrow{OE}$ shows only displacements which do produce elongations and consequently are controlled by elastic properties of the structural system

$$\mathbf{U}^e = \overrightarrow{OE}.$$

Finally, a new resolution of displacements is obtained

$$\mathbf{U} = \mathbf{U}^k + \mathbf{U}^e. \tag{64}$$

Now vectors $\mathbf{U}^k$ and $\mathbf{U}^e$ are mutually orthogonal and they have a clear physical sense. In the general case of $\mathfrak{R}^m$ vectors $\mathbf{U}^k$, $\mathbf{U}^e$ belong to two complement subspaces of $\mathfrak{R}^m$. The first one is matrix $\mathbf{A}_0^T$ nullspace and the second one is matrix $\mathbf{A}_0^T$ row space (Strang, 1988).

"Kinematic" and "elastic" displacements may be presented in the form

$$\mathbf{U}^k = z_1\mathbf{e}_1 + \cdots + z_{m-r}\mathbf{e}_{m-r} \tag{65}$$

$$\mathbf{U}^e = z_{m-r+1}\mathbf{e}_{m-r+1} + \cdots + z_m\mathbf{e}_m \tag{66}$$

or

$$\mathbf{U}^k = \mathbf{W}\mathbf{Z} \tag{67}$$

$$\mathbf{U}^e = \tilde{\mathbf{W}}\tilde{\mathbf{Z}} \tag{68}$$

$$\mathbf{W} = \{\mathbf{e}, \ldots, \mathbf{e}_{m-r}\}, \quad \mathbf{Z} = \{z_1, \ldots, z_{m-r}\}^T$$

$$\tilde{\mathbf{W}} = \{\mathbf{e}_{m-r+1}, \ldots, \mathbf{e}_m\}, \quad \tilde{\mathbf{Z}} = \{z_{m-r+1}, \ldots, z_m\}^T$$

here z_i is an unknown parameter; $\mathbf{e}_i$ is a vector of the nullspace basis ($i = 1, \ldots, m-r$) or $\mathbf{e}_i$ is a vector of the row space basis ($i = m-r+1, \ldots, m$).

3.1. *Statics*

Equilibrium equations in terms of displacements take the form

$$\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\mathbf{U}^e + \mathbf{D}\mathbf{U}^e + \mathbf{D}\mathbf{U}^k = \mathbf{Q} \tag{69}$$

or

$$\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}} + \mathbf{D}\mathbf{W}\mathbf{Z} = \mathbf{Q}. \tag{70}$$

This is a system of m scalar equations with m unknowns z_i. Figure 2 gives a simple

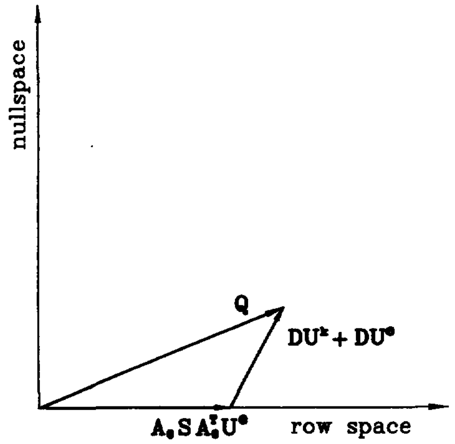


Fig. 2. Geometrical interpretation of eqn(69).

geometrical interpretation of eqn (69) in $\Re^2$. It is easy to observe that vector $\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\mathbf{U}^e$ lies in the row space. Indeed, this term can be interpreted as a linear combination of matrix $\mathbf{A}_0$ column-vectors or matrix $\mathbf{A}_0^T$ row-vectors. Consequently, this term does not give projection on to the nullspace.

Arbitrary vectors of the nullspace and the row space may be presented in the form

$$\mathbf{V} = \mathbf{W}\mathbf{Y} \tag{71}$$

$$\tilde{\mathbf{V}} = \tilde{\mathbf{W}}\tilde{\mathbf{Y}} \tag{72}$$

$$\mathbf{Y} = \{y_1, \ldots, y_{m-r}\}^T, \quad \tilde{\mathbf{Y}} = \{y_{m-r+1}, \ldots, y_m\}^T.$$

Multiplying scalarly eqn (70) by vectors $\mathbf{V}^T$, $\tilde{\mathbf{V}}^T$ it is possible to obtain correspondingly

$$\mathbf{Y}^T(\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}}+\mathbf{W}^T\mathbf{D}\mathbf{W}\mathbf{Z}) = \mathbf{Y}^T\mathbf{W}^T\mathbf{Q} \tag{73}$$

$$\tilde{\mathbf{Y}}^T(\tilde{\mathbf{W}}^T\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}\tilde{\mathbf{Z}}+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}}+\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\mathbf{Z}) = \tilde{\mathbf{Y}}^T\tilde{\mathbf{W}}^T\mathbf{Q}. \tag{74}$$

The physical sense of the projecting is the application of the principle of virtual work. Since eqns (73) and (74) should be satisfied under arbitrary vectors $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$, then the equations should be rewritten in the form

$$(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})\tilde{\mathbf{Z}}+\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\mathbf{Z} = \tilde{\mathbf{W}}^T\mathbf{Q} \tag{75}$$

$$\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\tilde{\mathbf{Z}}+\mathbf{K}^k\mathbf{Z} = \mathbf{W}^T\mathbf{Q} \tag{76}$$

where

$$\mathbf{K}^e = \tilde{\mathbf{W}}^T\mathbf{A}_0\mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}} \tag{77}$$

$$\mathbf{K}^k = \mathbf{W}^T\mathbf{D}\mathbf{W}. \tag{78}$$

After some transformations eqns (75) and (76) take the form

$$\{\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}-\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\}\tilde{\mathbf{Z}} = \{\tilde{\mathbf{W}}^T-\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T\}\mathbf{Q} \tag{79}$$

$$\{\mathbf{K}^k-\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\}\mathbf{Z} = \{\mathbf{W}^T-\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^T\}\mathbf{Q}. \tag{80}$$

By using the estimate

$$\|\mathbf{K}^e\| \approx \|\mathbf{S}\| \gg \|\mathbf{K}^k\| \approx \|\mathbf{D}\| \approx \|\mathbf{P}_0\| \tag{81}$$

and neglecting small values, eqns (79) and (80) take the form

$$\mathbf{K}^e\tilde{\mathbf{Z}} = \tilde{\mathbf{R}}\mathbf{Q} \tag{82}$$

$$\mathbf{K}^k\mathbf{Z} = \mathbf{R}\mathbf{Q} \tag{83}$$

where

$$\tilde{\mathbf{R}} = \tilde{\mathbf{W}}^T-\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}(\mathbf{K}^k)^{-1}\mathbf{W}^T \tag{84}$$

$$\mathbf{R} = \mathbf{W}^T-\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}(\mathbf{K}^e)^{-1}\tilde{\mathbf{W}}^T. \tag{85}$$

Thus, $\mathbf{K}^k$ is $m-r$ by $m-r$ "kinematic" stiffness matrix and $\mathbf{K}^e$ is r by r "elastic" stiffness matrix.

Member forces are obtained by the formula

$$\mathbf{P} = \mathbf{S}\mathbf{A}_0^T\mathbf{U}^e = \mathbf{S}\mathbf{A}_0^T\tilde{\mathbf{W}}(\mathbf{K}^e)^{-1}\tilde{\mathbf{R}}\mathbf{Q}. \tag{86}$$

Stability of the obtained stress strain state is provided by positive definiteness of the matrices $\mathbf{K}^e$ and $\mathbf{K}^k$. Matrix $\mathbf{K}^e$ is always positive definite thanks to its framework [it is implied that $\det(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}) \neq 0$, otherwise the initial equilibrium state is unstable]. Matrix $\mathbf{K}^k$ positive definiteness must be checked under initial configuration design.

Let the initial member forces be induced due to prestressing. In this case, solution of homogeneous eqn (1) is presented in the form

$$\mathbf{P}_0 = t_1\mathbf{p}_1 + \cdots + t_{n-r}\mathbf{p}_{n-r} \tag{87}$$

where vector $\mathbf{p}_i$ is the independent solution of homogeneous eqn (1) or matrix $\mathbf{A}_0$ nullspace basis vector (not to be confused with matrix $\mathbf{A}_0^T$ nullspace) and t_i is an unknown parameter. By using eqn (87) $\mathbf{D}$ takes the form

$$\mathbf{D} \equiv \mathbf{D}(\mathbf{P}_0) = t_1\mathbf{D}(\mathbf{p}_1) + \cdots + t_{n-r}\mathbf{D}(\mathbf{p}_{n-r}) \tag{88}$$

and

$$\mathbf{K}^k = t_1\mathbf{W}^T\mathbf{D}(\mathbf{p}_1)\mathbf{W} + \cdots + t_{n-r}\mathbf{W}^T\mathbf{D}(\mathbf{p}_{n-r})\mathbf{W} \tag{89}$$

and matrix $\mathbf{K}^k$ is considered as a function of $n-r$ parameters. The critical set of the parameters leads to the "kinematic" stiffness matrix singularity and indicates the critical prestressing forces. Thus, to provide stability of the initial state it is necessary to find the set of t_i which leads to the positive definite matrix $\mathbf{K}^k$. It may be effectively carried out by the Calladine–Pellegrino (Calladine and Pellegrino, 1991) algorithm.

3.2. *Instability: bifurcation of equilibrium*

The structure is in equilibrium if eqn (2) is satisfied. In the case of the bifurcation of the equilibrium state nonzero perturbations $\mathbf{A}'$, $\mathbf{P}'$ exist

$$(\mathbf{A}_0+\mathbf{A}+\mathbf{A}')(\mathbf{P}_0+\mathbf{P}+\mathbf{P}') = \mathbf{Q}_0+\mathbf{Q} \tag{90}$$

or expanding eqn (90) and omitting small values

$$\mathbf{A}_0\mathbf{P}'+\mathbf{A}'(\mathbf{P}_0+\mathbf{P}) = 0 \tag{91}$$

where

$$\mathbf{P}' = \mathbf{S}\mathbf{A}_0^T\,\delta\mathbf{U}^e \tag{92}$$

$$\mathbf{A}'(\mathbf{P}_0+\mathbf{P}) = \mathbf{D}'\delta(\mathbf{U}^e+\mathbf{U}^k) \tag{93}$$

$$\mathbf{D}' \equiv \mathbf{D}(\mathbf{P}_0)+\mathbf{D}(\mathbf{P}).$$

By this means equilibrium equations in terms of displacements can be obtained directly from eqns (75) and (76) with the help of replacements

$$\mathbf{Z}\rightarrow\delta\mathbf{Z},\quad \tilde{\mathbf{Z}}\rightarrow\delta\tilde{\mathbf{Z}},\quad \mathbf{D}\rightarrow\mathbf{D}',\quad \mathbf{Q}\rightarrow 0$$

which take the form

$$(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}})\delta\tilde{\mathbf{Z}}+\tilde{\mathbf{W}}^T\mathbf{D}'\mathbf{W}\delta\mathbf{Z}=0 \tag{94}$$

$$\mathbf{W}^T\mathbf{D}'\tilde{\mathbf{W}}\delta\tilde{\mathbf{Z}}+\hat{\mathbf{K}}^k\delta\mathbf{Z}=0 \tag{95}$$

where

$$\hat{\mathbf{K}}^k=\mathbf{W}^T\mathbf{D}'\mathbf{W}. \tag{96}$$

If external load $\mathbf{Q}$ grows proportionally to parameter ν then member forces may be presented in the form $\nu\,\mathbf{P}$ [see eqn (86)]. Consequently

$$\mathbf{D}'\equiv\mathbf{D}(\mathbf{P}_0)+\nu\mathbf{D}(\mathbf{P}).$$

Nonzero displacements appear under such values of parameter ν which lead to the equality

$$\det\begin{bmatrix}\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}} & \tilde{\mathbf{W}}^T\mathbf{D}'\mathbf{W}\\ \mathbf{W}^T\mathbf{D}'\tilde{\mathbf{W}} & \hat{\mathbf{K}}^k\end{bmatrix}=0. \tag{97}$$

Because of the estimate (81) the left upper submatrix elements are significantly larger than the rest. By using Laplace's resolution of determinant it is possible to approximately replace eqn (97) by the following one

$$\det(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}'\tilde{\mathbf{W}})\det\hat{\mathbf{K}}^k=0. \tag{98}$$

Again due to estimate (81) it is obvious that the minimum value of ν is obtained from the equation

$$\det\hat{\mathbf{K}}^k=0 \tag{99}$$

or

$$\det\{\mathbf{K}^k+\nu\mathbf{W}^T\mathbf{D}(\mathbf{P})\mathbf{W}\}=0. \tag{100}$$

3.3. *Vibrations*

In the case of vibrations, equilibrium equations (75) and (76) take the form

$$(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}})\tilde{\mathbf{Z}}+\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}\mathbf{Z}+\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}}\ddot{\tilde{\mathbf{Z}}}+\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W}\ddot{\mathbf{Z}}=\tilde{\mathbf{W}}^T\mathbf{Q} \tag{101}$$

$$\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}\mathbf{Z}+\mathbf{K}^k\mathbf{Z}+\mathbf{W}^T\mathbf{M}\tilde{\mathbf{W}}\ddot{\tilde{\mathbf{Z}}}+\mathbf{W}^T\mathbf{M}\mathbf{W}\ddot{\mathbf{Z}}=\mathbf{W}^T\mathbf{Q}. \tag{102}$$

Let the external load be presented in the form

$$\mathbf{Q}=\bar{\mathbf{Q}}\sin(\omega t+\vartheta) \tag{103}$$

then

$$\mathbf{U}^k=\bar{\mathbf{U}}^k\sin(\omega t+\vartheta)$$
$$\mathbf{U}^e=\bar{\mathbf{U}}^e\sin(\omega t+\vartheta) \tag{104}$$

and

$$\mathbf{Z}=\bar{\mathbf{Z}}\sin(\omega t+\vartheta)$$
$$\tilde{\mathbf{Z}}=\bar{\tilde{\mathbf{Z}}}\sin(\omega t+\vartheta) \tag{105}$$

where a bar over a letter designates amplitude value, ω is angular frequency, ϑ is the initial phase of vibrations.

Then, eqns (101) and (102) take the form

$$(\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}})\bar{\bar{\mathbf{Z}}}+(\tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W})\bar{\mathbf{Z}} = \tilde{\mathbf{W}}^T\bar{\mathbf{Q}} \tag{106}$$

$$(\mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}-\omega^2\mathbf{W}^T\mathbf{M}\tilde{\mathbf{W}})\bar{\bar{\mathbf{Z}}}+(\mathbf{K}^k-\omega^2\mathbf{W}^T\mathbf{M}\mathbf{W})\bar{\mathbf{Z}} = \mathbf{W}^T\bar{\mathbf{Q}}. \tag{107}$$

In the case of free vibrations $\bar{\mathbf{Q}} = 0$ and corresponding frequencies are obtained by using the equation

$$\det\begin{bmatrix}\mathbf{K}^e+\tilde{\mathbf{W}}^T\mathbf{D}\tilde{\mathbf{W}}-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}} & \tilde{\mathbf{W}}^T\mathbf{D}\mathbf{W}-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W} \\ \mathbf{W}^T\mathbf{D}\tilde{\mathbf{W}}-\omega^2\mathbf{W}^T\mathbf{M}\tilde{\mathbf{W}} & \mathbf{K}^k-\omega^2\mathbf{W}^T\mathbf{M}\mathbf{W}\end{bmatrix} = 0. \tag{108}$$

By using Laplace's resolution of the determinant and neglecting small values, eqn (108) takes the form

$$\det(\mathbf{K}^e-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}})\det(\mathbf{K}^k-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W}) = 0. \tag{109}$$

Because of the estimate (81), low frequencies are obtained by using condition

$$\det(\mathbf{K}^k-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W}) = 0 \tag{110}$$

and high frequencies are obtained by using condition

$$\det(\mathbf{K}^e-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}}) = 0. \tag{111}$$

If the forced vibrations frequency coincides with the free vibrations frequency then displacements increase unlimitedly. Otherwise, displacements values are obtained by using equations

$$\bar{\bar{\mathbf{Z}}} = (\mathbf{K}^e-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\tilde{\mathbf{W}})^{-1}\{\tilde{\mathbf{W}}^T-\tilde{\mathbf{W}}^T(\mathbf{D}-\omega^2\mathbf{M})\mathbf{W}(\mathbf{K}^k-\omega^2\mathbf{W}^T\mathbf{M}\mathbf{W})^{-1}\mathbf{W}^T\}\bar{\mathbf{Q}} \tag{112}$$

$$\bar{\mathbf{Z}} = (\mathbf{K}^k-\omega^2\mathbf{W}^T\mathbf{M}\mathbf{W})^{-1}\{\mathbf{W}^T-\mathbf{W}^T(\mathbf{D}-\omega^2\mathbf{M})\tilde{\mathbf{W}}(\mathbf{K}^e-\omega^2\tilde{\mathbf{W}}^T\mathbf{M}\mathbf{W})^{-1}\tilde{\mathbf{W}}^T\}\bar{\mathbf{Q}}. \tag{113}$$

4. NUMERICAL EXAMPLE

Consider the three-bars structure shown in Fig. 3. In this case

$$\mathbf{A}_0 = \begin{bmatrix} X_1 & X_1-X_3 & 0 \\ X_2 & X_2-X_4 & 0 \\ 0 & X_3-X_1 & X_3-2 \\ 0 & X_4-X_2 & X_4 \end{bmatrix} = \begin{bmatrix} -8 & -18 & 0 \\ 0 & 0 & 0 \\ 0 & 18 & 8 \\ 0 & 0 & 0 \end{bmatrix}. \tag{114}$$

The matrix elements are coordinates differences, but not direction cosines which are more suitable. Then the real member forces and elongations are obtained by using the formula

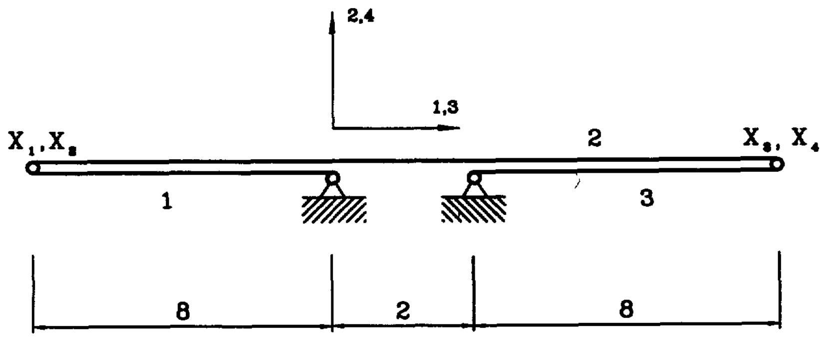


Fig. 3. Three-bars assembly.

$$P_i^r = P_i l_i, \quad \Delta_i^r = \Delta_i / l_i \tag{115}$$

where l_i is the ith member length.

Matrix $\mathbf{A}_0$ rank is 2 and the structure possesses self stress state.

Let us find matrix $\mathbf{A}_0$ nullspace basis vectors $\mathbf{p}_i$ and matrix $\mathbf{A}_0^T$ nullspace and row space basis vectors $\mathbf{e}_i$.

$\mathbf{p}_i$ is obtained by solving equation

$$\mathbf{A}_0 \boldsymbol{\zeta} = 0 \tag{116}$$

or

$$\begin{aligned} -8\zeta_1 - 18\zeta_2 - 0\zeta_3 &= 0 \\ 0\zeta_1 + 18\zeta_2 + 8\zeta_3 &= 0. \end{aligned} \tag{117}$$

These are two independent equations with three unknowns. Assuming ζ_2 to be independent it is obtained that

$$\begin{aligned} \zeta_1 &= -\frac{18}{8}\zeta_2 \\ \zeta_2 &= 1\zeta_2 \\ \zeta_3 &= -\frac{18}{8}\zeta_2 \end{aligned} \tag{118}$$

or

$$\zeta = \begin{bmatrix} -\frac{18}{8} \\ 1 \\ -\frac{18}{8} \end{bmatrix} \zeta_2 = \begin{bmatrix} -9 \\ 4 \\ -9 \end{bmatrix} \times \text{const} = \mathbf{p}_1 \times \text{const}. \tag{119}$$

In the same manner the following equation is solved

$$\mathbf{A}_0^T \boldsymbol{\xi} = 0 \tag{120}$$

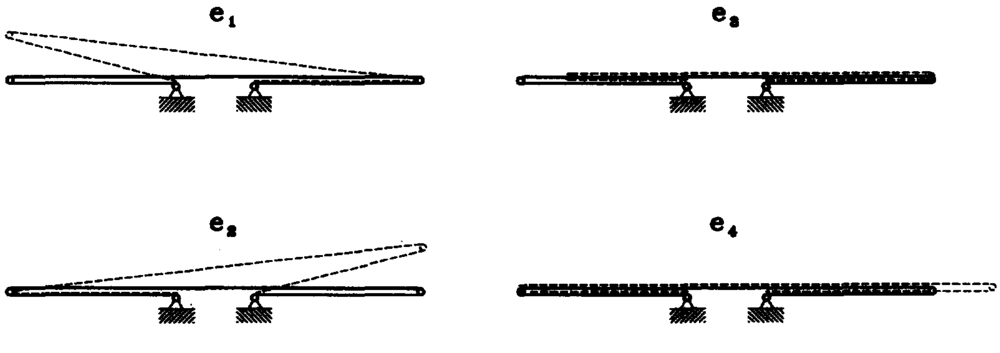
or

$$\begin{aligned} -8\xi_1 + 0\xi_2 + 0\xi_3 + 0\xi_4 &= 0 \\ 0\xi_1 + 0\xi_2 + 8\xi_3 + 0\xi_4 &= 0. \end{aligned} \tag{121}$$

These are two independent equations with four unknowns. Assuming ξ_2, ξ_4 to be independent it is possible to obtain that

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 &= \xi_2 \\ \xi_3 &= 0 \\ \xi_4 &= \xi_4 \end{aligned} \tag{122}$$

or



a. b.

Fig. 4. "Kinematic" and "elastic" modes of displacements.

$$\xi = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \xi_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \xi_4 = \mathbf{e}_1 \xi_2 + \mathbf{e}_2 \xi_4. \tag{123}$$

On the other hand eqns (121) may be rewritten as

$$\mathbf{e}_3^{\mathrm{T}} \boldsymbol{\xi} = 0$$

$$\mathbf{e}_4^{\mathrm{T}} \boldsymbol{\xi} = 0. \tag{124}$$

Finally

$$\mathbf{p}_1 = \begin{bmatrix} -9 \\ 4 \\ -9 \end{bmatrix}; \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{125}$$

As it was shown before $\mathbf{p}_1$ presents the mode of the prestressing forces; $\mathbf{e}_1$, $\mathbf{e}_2$ present the modes of "kinematic" displacements [Fig. 4(a)]; $\mathbf{e}_3$, $\mathbf{e}_4$ present the modes of "elastic" displacements [Fig. 4(b)].

Stability of a self stress state is checked by calculating matrix $\mathbf{D}(\mathbf{P}_0)$ by using the equality

$$\mathbf{A}\mathbf{P}_0 = \mathbf{D}(\mathbf{P}_0)\mathbf{U}. \tag{126}$$

It is possible to obtain that

$$\mathbf{A}\mathbf{P}_0 = \mathbf{A}\mathbf{p}_1 t_1 = \begin{bmatrix} U_1 & U_1 - U_3 & 0 \\ U_2 & U_2 - U_4 & 0 \\ 0 & U_3 - U_1 & U_3 \\ 0 & U_4 - U_2 & U_4 \end{bmatrix} \begin{bmatrix} -9 \\ 4 \\ -9 \end{bmatrix} t_1 = \begin{bmatrix} -5U_1 - 4U_3 \\ -5U_2 - 4U_4 \\ -4U_1 - 5U_3 \\ -4U_2 - 5U_4 \end{bmatrix} t_1$$

$$= t_1 \begin{bmatrix} -5 & 0 & -4 & 0 \\ 0 & -5 & 0 & -4 \\ -4 & 0 & -5 & 0 \\ 0 & -4 & 0 & -5 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}. \tag{127}$$

Thus,

$$\mathbf{D}(\mathbf{P}_0) = t_1 \begin{bmatrix} -5 & 0 & -4 & 0 \\ 0 & -5 & 0 & -4 \\ -4 & 0 & -5 & 0 \\ 0 & -4 & 0 & -5 \end{bmatrix}. \tag{128}$$

By using eqns (67), (125) and (128), eqn (78) takes the form

$$\mathbf{K}^k = t_1 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 & -4 & 0 \\ 0 & -5 & 0 & -4 \\ -4 & 0 & -5 & 0 \\ 0 & -4 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = t_1 \begin{bmatrix} -5 & -4 \\ -4 & -5 \end{bmatrix}. \tag{129}$$

$\mathbf{K}^k$ eigenvalues are $-9t_1$, $-1t_1$. It means that the self stress state stiffens the structure under negative values of t_1. Let $t_1 = -5$, then

$$\mathbf{P}_0 = -5\mathbf{p}_1 = \begin{bmatrix} 45 \\ -20 \\ 45 \end{bmatrix}. \tag{130}$$

The real initial member forces are

$$\mathbf{P}_0^r = \begin{bmatrix} 360 \\ -360 \\ 360 \end{bmatrix}. \tag{131}$$

Thus, members 1 and 3 are tensioned by the forces of magnitude 360 and member 2 is compressed by the force of magnitude 360.

In the case where the structure is loaded as it is shown in Fig. 5.

$$\mathbf{Q} = \{1, 0, 0, 0\}^T. \tag{132}$$

By using eqns (129) and (130) and taking into account $t_1 = 5$, eqn (83) takes the form

$$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{133}$$

It means that the load does not cause "kinematic" displacements.

"Elastic" displacements and member forces are obtained by using eqns (82), (84), (86) and (68)

Fig. 5. Stability problem.

$$\mathbf{U}^e = \frac{1}{ES}\begin{bmatrix} 6.11765 \\ 0 \\ 1.88235 \\ 0 \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} -0.0955882 \\ -0.0130719 \\ 0.0294118 \end{bmatrix} \tag{134}$$

where E is Young modulus and S is square of member cross-section.

The real member forces are

$$\mathbf{P}^r \cong \begin{bmatrix} -0.764706 \\ -0.235294 \\ 0.235294 \end{bmatrix}. \tag{135}$$

Now consider a bifurcation of the equilibrium.

First matrix **D(P)** is calculated from equation

$$\mathbf{A}'\mathbf{P} = \mathbf{D}(\mathbf{P})\delta\mathbf{U}. \tag{136}$$

It is obtained

$$\mathbf{D}(\mathbf{P}) = \begin{bmatrix} -0.10866 & 0 & 0.0130719 & 0 \\ 0 & -0.10866 & 0 & 0.0130719 \\ 0.0130719 & 0 & 0.0163399 & 0 \\ 0 & 0.0130719 & 0 & 0.0163399 \end{bmatrix}. \tag{137}$$

Finally, eqn (100) takes the form

$$\det\left\{\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} + v\begin{bmatrix} -0.010866 & 0.0130719 \\ 0.0130719 & 0.0163399 \end{bmatrix}\right\} = 0. \tag{138}$$

By solving this equation it is possible to find the critical value of the external load

$$v_{cr} \cong 75.5556$$

$$\mathbf{Q}^{cr} = \{75.5556, 0, 0, 0\}^T. \tag{139}$$

The considered numerical example illustrates the features of the "kinematic/elastic" approach of the resolution of displacements which seems to be more abstract. If the "natural" approach is used then it is suitable to change numeration of degrees of freedom

$$X_2 \leftrightarrow X_3$$

and consequently to change the second and third rows of the equilibrium matrix

$$\mathbf{A}_0 = \begin{bmatrix} X_1 & X_1 - X_2 & 0 \\ 0 & X_2 - X_1 & X_2 - 1 \\ X_3 & X_3 - X_4 & 0 \\ 0 & X_4 - X_3 & X_4 \end{bmatrix} = \begin{bmatrix} -8 & -18 & 0 \\ 0 & 18 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{140}$$

By this means

$$\mathbf{A}_{011} = \begin{bmatrix} -8 & -18 \\ 0 & 18 \end{bmatrix}; \quad \mathbf{A}_{012} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$\mathbf{A}_{021} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{A}_{022} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently, using eqns (15) it is obtained that

$$\mathbf{T} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{141}$$

and displacements resolution takes the form

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{U}_2^h \end{bmatrix} + \begin{bmatrix} \mathbf{U}_1^p \\ 0 \end{bmatrix}. \tag{142}$$

In other words, "natural" resolution of displacements coincides with the "kinematic" and "elastic" one for this example.

Perturbated equilibrium matrix allows us to obtain (omitting details)

$$\mathbf{D}(\mathbf{P}_0) = \begin{bmatrix} -5 & -4 & 0 & 0 \\ -4 & -5 & 0 & 0 \\ 0 & 0 & -5 & -4 \\ 0 & 0 & -4 & -5 \end{bmatrix} t_1$$

$$\mathbf{D}_{11}(\mathbf{P}_0) = \mathbf{D}_{22}(\mathbf{P}_0) = \begin{bmatrix} -5 & -4 \\ -4 & -5 \end{bmatrix} t_1$$

$$\mathbf{D}_{12}(\mathbf{P}_0) = \mathbf{D}_{21}(\mathbf{P}_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} t_1$$

$$t_1 = -5 \tag{143}$$

$$\mathbf{D}(\mathbf{P}) = \begin{bmatrix} -0.10866 & 0.0130719 & 0 & 0 \\ 0.0130719 & 0.0163399 & 0 & 0 \\ 0 & 0 & -0.10866 & 0.0130719 \\ 0 & 0 & 0.0130719 & 0.0163399 \end{bmatrix}$$

$$\mathbf{D}_{11}(\mathbf{P}) = \mathbf{D}_{22}(\mathbf{P}) = \begin{bmatrix} -0.10866 & 0.013079 \\ 0.013079 & 0.0163399 \end{bmatrix}$$

$$\mathbf{D}_{12}(\mathbf{P}) = \mathbf{D}_{21}(\mathbf{P}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{144}$$

and the "natural" approach leads to the same results.

5. DISCUSSION

5.1. *Statics*

Equations (82)–(85) represent the linear statics of underconstrained structures and lead to some interesting qualitative conclusions:

(i) Generally external load produces both "kinematic" and "elastic" displacements, even though it lies in the column space of the initial equilibrium matrix ($\mathbf{W}^T\mathbf{Q} = 0$) or, on the contrary, if it lies in the orthogonal complement space to the column space ($\tilde{\mathbf{W}}^T\mathbf{Q} = 0$).

(ii) If the external load lies neither in the column space of the initial equilibrium matrix nor close to it, then the second term on the right-hand side of eqn (85) can be neglected and magnitudes of "kinematic" displacements are *significantly larger* than those of "elastic" ones.

(iii) If the external load lies in the column space of the initial equilibrium matrix or close to it, then the second term on the right-hand side of eqn (85) dominates and can not be neglected. In this case "kinematic" and "elastic" displacements are approximately of the same magnitudes.

5.2. *Stability*

Equation (100) represents the problem of bifurcation of equilibrium for underconstrained structures. It is interesting to compare this equation to the analogous one for conventional fully constrained structures

$$\det\{\mathbf{K}+\nu\mathbf{D}(\mathbf{P})\} = 0 \tag{145}$$

where $\mathbf{K}$ is defined by eqn (8).

The magnitudes of elements of matrix $\mathbf{K}$ which depend upon elastic properties of the members are significantly larger than the ones of matrix $\mathbf{K}^k$ which depend only upon the values of the initial member forces. As a consequence of this, the magnitude of the critical parameter ν_{cr} depends upon the elastic properties of the members for fully constrained structures and does not depend upon them for underconstrained structures. It is necessary to note that the increments of the member forces included in the second terms in the braces of eqns (100) and (145) depend upon correlations of the members stiffnesses, but not upon absolute values of the stiffnesses. Thus, the influence of the elastic properties of the members on the critical bifurcation load is negligible for underconstrained structures in contrast to the fully constrained ones.

5.3. *Vibrations*

The frequencies of free vibrations are defined by eqns (110) and (111). Considering things in the same manner as in the previous subsection it is possible to conclude that the low and most dangerous frequencies are affected by prestressing and the high ones are affected by elastic properties of the structure.

It is interesting to note that a classical string can be approximated by a system of numerous pin-jointed members. It can be easily obtained that "kinematic" displacements are transverse and the "elastic" ones are longitudinal. Low frequencies of free vibrations are associated with "kinematic" transverse displacements and high frequencies are associated with "elastic" longitudinal ones which correspond with well-known results for continuous string.

6. CONCLUDING REMARKS

(1) Rigidity of traditional fully constrained structures is provided by elastic stiffness of the structures. For nontraditional underconstrained structures elastic stiffness is partly replaced by "kinematic", due to prestressing (initial), stiffness. This replacement allows us to remove some structural members or, in other words, to lighten the structure significantly. This is an important advantage of underconstrained structures. On the other hand the magnitude of the "kinematic" stiffness is lower than the "elastic" stiffness. This leads to the following features of the underconstrained structures:

—in general "elastic" displacements are small in comparison to "kinematic" ones;

—the magnitude of the critical load in the stability problem is affected essentially by prestressing forces (or initial member forces);

—the magnitude of the lowest frequency of free vibrations is affected essentially by pre-stressing forces (or initial member forces).

(2) The considered theory of underconstrained structures reduces to a traditional analysis of pin-jointed bars in a particular case where $m = r$. In this sense the presented theory is an extension of the classical concepts.

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